Clairaut's Form

A differential equation of the form

$$y = px + f(p), \quad p = \frac{dy}{dx} \quad \text{— (1)}$$

is called Clairaut's equation.

It's complete primitive is

$$y = cx + f(c)$$

Justification:→

Differentiate the given equation (1) w.r.to x,

$$p = p + \{x + f'(p)\}\frac{dp}{dx}$$

$$\Rightarrow \{x + f'(p)\}\frac{dp}{dx} = 0$$

This gives either $\frac{dp}{dx} = 0$ — (2) or $x + f'(p) = 0$ — (3)

(2) gives $p = c$ — (4)

From (1) & (4) we eliminate p and obtain the complete primitive in the form

$$y = cx + f(c)$$

Geometrically this represents a family of straight lines.

Now from (1) & (3)

$$\left.\begin{aligned} x &= -f'(p) \\ y &= -pf'(p) + f(p) \end{aligned}\right\} \text{— (5)}$$

We see at once that it contains no arbitrary constant and therefor is not a general solution. But it is a solution of ① as can be easily verified. In fact ⑤ gives a singular solution.

Example ① :→ Solve $y = px + \frac{a}{p}$ — ①

Solution:→ Differentiating ① w.r.to x

$$\therefore p = p + x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$$

Either $\frac{dp}{dx} = 0$ or $x - \frac{a}{p^2} = 0$

The first gives $p = c$ & the general solution is $y = cx + \frac{a}{c}$

second gives $x = \frac{a}{p^2}$ — ②

$$\therefore y = px + \frac{a}{p}$$

$$= p \cdot \frac{a}{p^2} + \frac{a}{p}$$

$$\therefore y = \frac{2a}{p} \quad — ③$$

Eliminating p from ② & ③, we obtain $y^2 = 4ax$. is the singular solution of ①

Example (2) :→ Solve $y = px + p - p^2$ — (1)

Solution:→

The given equation is in clairaut's form

Hence general solution is $y = cx + c - c^2$

where c is a parameter.

Differentiating (1), w.r.to x

$$p = p + x\frac{dp}{dx} + \frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\Rightarrow (x + 1 - 2p)\frac{dp}{dx} = 0$$

If $x + 1 - 2p = 0$

$$\Rightarrow 2p = x + 1$$

$$\Rightarrow p = \frac{x+1}{2}$$

putting this value of p in (1)

$$y = \frac{x+1}{2}x + \frac{x+1}{2} - \left(\frac{x+1}{2}\right)^2$$

$$\Rightarrow y = \frac{x(x+1)}{2} + \frac{(x+1)}{2} - \frac{(x+1)^2}{4}$$

$$= \frac{2x(x+1) + 2(x+1) - (x+1)^2}{4}$$

$$\Rightarrow 4y = 2(x+1)^2 - (x+1)^2$$

$$\Rightarrow 4y = (x+1)^2$$

∴ The singular solution is $4y = (x+1)^2$ Ans.